

A Pragmatic Interpretation of Quantum Logic

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Abstract

Scholars have wondered for a long time whether the language of quantum mechanics introduces a quantum notion of truth which is formalized by quantum logic (QL) and is incompatible with the classical (Tarskian) notion. We show that QL can be interpreted as a pragmatic language \mathcal{L}_Q^P of *assertive* formulas, which formalize statements about physical systems that are empirically *justified* or *unjustified* in the framework of quantum mechanics. According to this interpretation, QL formalizes properties of the metalinguistic notion of *empirical justification within quantum mechanics* rather than properties of a quantum notion of truth. This conclusion agrees with a general integrationist perspective that interprets non-standard logics as theories of metalinguistic notions different from truth, thus avoiding incompatibility with classical notions and preserving the globality of logic. By the way, some elucidations of the standard notion of quantum truth are also obtained.

Key words: pragmatics, quantum logic, quantum mechanics, justifiability, global pluralism.

1 Introduction

Several years ago a *formalized pragmatic calculus* \mathcal{L}^P was constructed, based on some deep ideas by Dalla Pozza [Dalla Pozza and Garola 1995]. This calculus extends a classical propositional calculus stemming from Frege's *ideographic language*, in which the *assertion sign* is introduced as a constitutive part of the formulas of the logical calculus [Frege, 1879, 1891, 1893, 1918; Reichenbach, 1947; Stenius, 1969]. \mathcal{L}^P is obtained indeed by considering a classical propositional calculus (CPC) with standard connectives and formation rules for *radical formulas*, and adding, besides the assertion sign, *pragmatic connectives* and formation rules for *assertive formulas*. The radical formulas of \mathcal{L}^P are then supplied with a classical semantic interpretation, while the assertive formulas are

supplied with a pragmatic interpretation in terms of the notion of *justification* (or *proof*).

The first aim of this construction is providing a general framework in which the conflict between the classical and the verificationist theories of truth and meaning can be settled by integrating their perspectives. The verificationist theories of truth, indeed, have been criticized by many scholars, mainly because they assume that a proposition is true if and only if asserting it is justified, which leads to identify the notions of truth and justification [Russel, 1940, 1950; Carnap, 1949; Popper, 1969; Haack, 1978]. There are strong intuitive arguments that support the need of avoiding this identification. In fact, the pragmatic notion of justification presupposes the semantic notion of truth, for a proof of a sentence consists in showing that the truth value of the sentence is *true*. Moreover, there are factual and logico-mathematical sentences that are *undecidable*, i.e., epistemically inaccessible, though they have a truth value [Carnap, 1932; Russell, 1940]. A sharp distinction between justification and truth is therefore introduced in \mathcal{L}^P via the assertion sign, which has a purely pragmatic role and cannot be identified with an alethic modality operator (it formalizes indeed the metalinguistic notion of proof, which cannot be reduced to any acceptable notion of truth, in the object language). Correspondingly, there are deep syntactic differences between the part of \mathcal{L}^P formalizing the properties of the classical notion of truth (radical formulas) and the part of \mathcal{L}^P formalizing the general properties of the notion of justification (assertive formulas).

The second aim of the construction of \mathcal{L}^P is showing that the integration of the notions of truth and justification realized in \mathcal{L}^P allows one to settle, in particular, the conflict between classical and intuitionistic logic in a unified perspective. To this end two partially overlapping structures, ACPC and AIPC (where A stands for *assertive*) are singled out in \mathcal{L}^P that are isomorphic to CPC and to an intuitionistic propositional calculus (IPC), respectively. The pragmatic interpretation of \mathcal{L}^P induces, through the isomorphism of AIPC and IPC, an interpretation of IPC which recovers in a natural way the standard Brower-Heyting-Kolmogorov (BHK) interpretation of this calculus in terms of logical proof [Troelstra and Van Dalen, 1988]. The construction of \mathcal{L}^P can thus help enlightening “the mysteries of the intuitionistic truth” [Van Dalen, 1986].

The general perspective underlying the construction of \mathcal{L}^P is resumed in the Introduction of the paper by Dalla Pozza and Garola as follows.

“The purpose of our interpretation is mainly philosophical. Indeed we aim to settle the conflicts between classical and intuitionistic logic, and between the classical (correspondence) and the intuitionistic (verificationist) conceptions of truth and meaning (see Dummett, 1977, 1978, 1979, 1980; Prawitz, 1977, 1980, 1987); this will be done by introducing an *integrated* perspective which preserves both the *globality* of logic (in the sense of the *global pluralism*, which admits the existence of a plurality of mutually compatible logical systems, but not of systems which are mutually incompatible or rivals, see Haack 1978, Chapter 12) and the *classical* notion of *truth as corre-*

spondence, which we may consider *explicated* rigorously by Tarski’s semantic theory (see Tarski 1933, 1944).”

The ideas underlying the construction of \mathcal{L}^P have started a lively debate on pragmatics and related topics. Based on Dalla Pozza’s distinction between descriptive and expressive notions of norms, Bellin and Dalla Pozza [2002], Bellin and Ranalter [2003] and Ranalter [2006] have developed a pragmatic theory of obligations, assertions and causal implication; White [2008] has presented a formal theory of actions; Carrara and Chiffi [2013] have applied the logic for pragmatics framework to knowability paradoxes. Moreover intuitionistic dualities have been explored from the viewpoint of the logic for pragmatics, where co-intuitionism has been regarded as a logic of hypotheses in relation with the intuitionistic logic of assertions [Bellin and Biasi 2004; Biasi and Aschieri 2008; Bellin, 2014; Bellin 2015; Bellin et al., 2015a; Bellin et al., 2015b].

The quotation above from the paper by Dalla Pozza and Garola also reminds us that there are several research fields in which non-classical notions of truth are introduced, raising conflicts with classical logic (CL) similar to the conflict between intuitionistic logic and CL. A typical example is provided by quantum logic (QL), with its non-classical structure that is claimed to imply a highly problematical notion of *quantum truth* (a huge literature exists on this topic; the interested reader can find a general review of the attempts at constructing a logic for the language of quantum mechanics till the early seventies in the classical book by Jammer [1974], and more updated treatments and bibliographies in the books by Redei [1998] and Dalla Chiara *et al.* [2004]). One can then wonder whether some of the foregoing conflicts can be settled by resorting to \mathcal{L}^P and embedding into it the non-classical structures that are considered, as in the case of intuitionistic logic. The answer is positive, as such an embedding has been realized for Girard’s *linear logic* [Girard, 1987] by Bellin and Dalla Pozza [2002]. We aim to show in this paper that a similar result can be achieved if QL is considered. The general scheme provided by \mathcal{L}^P applies indeed not only when the notion of proof is specified to be a *logical proof*, as in the case of IPC, but also if it is specified to be an *empirical proof* (or *verification*, or *empirical justification*) in the framework of a specific theory.

Let us resume the main lines of our work.

By considering the notion of truth in quantum mechanics, we observe in Sect. 2 that our program has to face a deep problem from the very beginning. Indeed, truth and verification are strictly entangled in the standard interpretations of quantum mechanics. To avoid this problem we introduce a generalization of \mathcal{L}^P in Sect. 3 (still denoted by \mathcal{L}^P , by abuse of language) admitting a *partial classical semantics* for radical formulas. This semantics can be particularized to fit in with various different interpretations of quantum mechanics. If one adopts a suitable orthodox interpretation or a modal interpretation, our notion of truth weakens the classical notion but does not conflict with it. If one accepts the generalization and reinterpretation of quantum mechanics (*extended semantic realism*, or *ESR, model*) proposed by us together with some collaborators [Garola, 2015; Garola and Sozzo, 2009, 2010, 2011a, 2011b, 2011c, 2012; Garola

and Persano, 2014; Garola *et al.*, 2014, 2015], our notion of truth coincides with the classical notion. Based on this generalization of \mathcal{L}^P , we select in Sect. 4 a sublanguage \mathcal{L}_Q^P of \mathcal{L}^P in which the notion of proof is specified to be the notion of empirical proof in quantum mechanics. We then show in Sect. 5 that this specification induces a homomorphism of \mathcal{L}_Q^P onto QL: hence, an interpretation of QL as a structure formalizing the properties of the metalinguistic notion of justification according to quantum mechanics, not the properties of a notion of truth alternative to the classical notion.¹ This result holds for each of the interpretations of QM mentioned above (orthodox, modal and ESR model), hence we can claim that our aim has been reached.

We conclude our work in Sect. 6 by comparing our result with a similar result obtained by ourselves together with another author [Garola and Sozzo, 2013], showing the advantages of the approach proposed in this paper.

To close, we note that our generalization of \mathcal{L}^P is important also independently of quantum mechanics, because it makes \mathcal{L}^P a more powerful tool for coping with a variety of problems.

2 On the notion of truth in quantum mechanics

Busch *et al.* [1991, 1996] distinguish two basic classes of interpretations of quantum mechanics, the class of *statistical* and the class of *realistic* interpretations. According to the statistical interpretations quantum mechanics deals only with probabilities of measurement outcomes, and no reference to single items of physical systems (briefly, *individual objects* in the following) is allowed. According to the realistic interpretations quantum mechanics deals with individual objects and their physical properties.

The statistical interpretations imply an instrumentalist view that has been severely criticized from an epistemological viewpoint. For instance, Timpson [2006] writes:

“The point is, instrumentalism is not a particularly attractive or interesting interpretive option in quantum mechanics, amounting more to a refusal to ask questions than to take quantum mechanics seriously. It is scarcely the epistemologically enlightened position that older generations of physicists, suffering from positivistic hangovers, would have us believe.”

We add that nowadays experimental physicists often claim that they can deal with individual objects, not only with statistical ensembles.

¹This procedure does not strictly match the procedure adopted in [Dalla Pozza and Garola, 1995] to recover intuitionistic logic within \mathcal{L}^P . Indeed the axioms of AIPC (which make AIPC isomorphic to IPC) are sentences of \mathcal{L}^P that are *pragmatically valid* (*p-valid*) in \mathcal{L}^P , for they characterize a notion of logical proof. Some of the axioms that characterize the quantum notion of proof are instead sentences of \mathcal{L}_Q^P that are not p-valid. We therefore avoid a purely axiomatic approach, which would make our task uselessly complicate.

The class of the realistic interpretations, on the other side, is very broad. In fact, the requirement that quantum mechanics deals with individual objects characterizes a very weak form of realism, which does not imply any wave or particle model for individual objects nor necessarily entails ontological commitments (one could indeed interpret the term “individual object” as “activation of a preparing procedure” [Ludwig, 1983]). Hence this class includes both standard interpretations and reinterpretations/modifications of quantum mechanics that are realistic in a stronger sense (as Bohm’s theory, many worlds interpretation, GRW theory, etc.). To avoid misunderstandings we therefore call the interpretations of this class *individual* rather than *realistic* interpretations in the following.

Whenever an individual interpretation is adopted, however, the measurement problem arises, which was known since the birth of quantum mechanic and is clearly formalized by some famous “no-go” theorems, as Bell-Kochen-Specker’s [Bell, 1966; Kochen and Specker, 1967] and Bell’s [Bell, 1964], stating the *contextuality* and the *nonlocality* (i.e., contextuality at a distance) of quantum mechanics, respectively (see, e.g., [Mermin, 1993]). In short, the result of a measurement of a physical property on an individual object in a quantum state² is not prefixed according to quantum mechanics, but it depends on the set of (compatible) measurements that are simultaneously performed on the individual object (even at a distance, in the case of measurements performed on an individual object that is an item of a composite system whose component parts are far away). Of course, this typical feature of quantum mechanics does not follow because of flaws or errors in the measurement devices (the measurements are supposed to be *exact*). Hence, generally, no truth value can be assigned to sentences attributing physical properties to individual objects in a given state without taking into account the set of measurements that are performed. Thus, truth and verification by means of measurements could not be separated in quantum mechanics, whose language would require the adoption of a (non-classical) verificationist theory of truth.

The feature of quantum mechanics described above raises a deep problem, as we have anticipated in Sect. 1. It implies in fact that the general scheme provided by the original formulation of \mathcal{L}^P does not fit in with an individual interpretation of quantum mechanics, because no truth assignment could be

²We recall that “quantum states” and “physical properties” can be considered as theoretical terms of the language of quantum mechanics which can be operationally interpreted as follows. A quantum state S^Q corresponds to a subset of physically equivalent *preparation procedures* belonging to a set of preparation procedures associated with Ω . A physical property E corresponds to a subset of physically equivalent *dichotomic registering devices* belonging to a set of dichotomic registering devices associated with Ω . Each device r of the latter subset, if activated in succession with the activation of a preparation procedure p in the subset corresponding to S^Q , performs a measurement of E on an item of Ω prepared by p , that is, on an *individual object a in (the quantum state) S^Q* , after which a either displays E or not. In this sense we say that E is *testable* [Beltrametti and Cassinelli, 1981; Ludwig, 1983; Garola and Sozzo 2014] (we remind, however, that there are physical properties in quantum mechanics that are *incompatible*, in the sense that they cannot be tested conjointly). We add that quantum states are usually divided into two disjoint classes, the class of *pure* quantum states and the class of *mixed* quantum states, or *mixtures*.

defined on the set of radical formulas of \mathcal{L}^P (whenever these formulas are interpreted as sentences of the language of quantum mechanics) independently of the justification value of the corresponding assertive formulas. However, we maintain that this problem can be avoided by introducing a suitable generalization of \mathcal{L}^P . Indeed, for every quantum state S^Q of a physical system there are observables whose values can be predicted with probability 1, independently of the measurement context (to be precise, all observables that admit S^Q as an eigenstate). Hence one can associate with S^Q a subset of *objective* properties, that is, a subset of physical properties that are possessed with probability 1 or 0 by every individual object a whose state is S^Q , independently of any measurement. One can then assign a truth value that does not depend on the measurement context to each sentence attributing one of these physical properties to a : truth value *true* if the probability is 1, truth value *false* if the probability is 0. Thus one obtains a noncontextual *quantum partial truth assignment* associated with S^Q . Hence truth and verification can be distinguished if one restricts to the subset of objective properties.

Bearing in mind the above argument, we will generalize \mathcal{L}^P by admitting truth assignments on radical formulas that are not defined everywhere (the *partial classical semantics* mentioned in Sect. 1). This generalization fits in well with those standard interpretations of quantum mechanics that consider *real* (or *actual*) in the quantum state S^Q every physical property E such that the probability $p(S^Q, E)$ associated by quantum mechanics to the pair (S^Q, E) is 1 [Jauch, 1968; Piron, 1976; Busch *et al.*, 1991, 1996; Aerts, 1999]. We call these interpretations *standard realistic interpretations* in the following, because their proposers usually maintain that they express a realistic philosophical position.

Our generalization of \mathcal{L}^P fits in well also with other interpretations of quantum mechanics that introduce new sets of states besides quantum states. We refer in particular to the *modal interpretations* of quantum mechanics (see the bibliography in [Lombardi and Dieks, 2014]). In these interpretations the *dynamical states* correspond to the quantum states. But a further set of *value states* is introduced, and the set of all sentences attributing a physical property to an individual object a that have a truth value is determined by the value state of a . This truth assignment is consistent with the quantum partial truth assignment introduced above but it is defined on a broader set of sentences. However, every truth assignment associated with a value state is still partial, and can be seen as an instantiation of the generalized semantics introduced in \mathcal{L}^P .

Finally, our generalization of \mathcal{L}^P is compatible also with the ESR model proposed by ourselves together with some collaborators in the papers quoted in Sect. 1. This model in fact generalizes quantum mechanics and circumvents the “no-go” theorems by reinterpreting quantum probabilities as conditional on detection rather than absolute. Contextuality and nonlocality are thus avoided. Hence, truth values are defined for every quantum state and for all sentences of the language of quantum mechanics according to classical rules and independently of the measurement context. This truth assignment is a borderline case of the quantum partial truth assignment introduced above because it is defined everywhere. Hence it can be considered as an instantiation of the semantics

provided in the original version of \mathcal{L}^P , no generalization being needed. It is interesting, however, to observe that one can construct *hidden variables models* for the ESR model in which *microscopic states* are introduced that play a role analogous to the role of value states in the modal interpretations: the main difference is that the truth assignments associated with the microscopic states are not partial but defined everywhere [Garola et al., 2015].

3 The generalized pragmatic language \mathcal{L}^P

Let us summarize syntax, semantics and pragmatics of a pragmatic language \mathcal{L}^P which generalizes the language denoted by the same symbol in the paper by Dalla Pozza and Garola [1995].

Alphabet. The alphabet \mathcal{A}^P of \mathcal{L}^P contains as *descriptive signs* the propositional letters p, q, r, \dots ; as *logical-semantic signs* the connectives $\neg, \wedge, \vee, \rightarrow$ and \leftrightarrow ; as *logical-pragmatic signs* the assertion sign \vdash and the connectives N, K, A, C and E ; as *auxiliary signs* the round brackets $(,)$.

Radical formulas. The set ψ_R of all *radical formulas (rfs)* of \mathcal{L}^P is made up by all formulas constructed by means of descriptive and logical-semantic signs, following the standard recursive rules of classical propositional logic. We denote by ϕ_R the subset of all rfs consisting of a propositional letter only (*atomic formulas*).

Assertive formulas. The set ψ_A of all *assertive formulas (afs)* of \mathcal{L}^P is made up by all rfs preceded by the assertive sign \vdash (*elementary afs*), plus all formulas constructed by using elementary afs and following standard recursive rules in which N, K, A, C and E take the place of $\neg, \wedge, \vee, \rightarrow$ and \leftrightarrow , respectively. We denote by ϕ_A the subset of all elementary afs of ψ_A .

Semantic interpretation. Let us introduce a family $\{\sigma_S\}_{S \in \mathcal{S}}$, where \mathcal{S} is a set of *states* which play the role of possible worlds in Kripkean semantics and σ_S is a function which maps a subset ϕ_{RS} of ϕ_R (the *domain* of σ_S) onto the set $\{1, 0\}$ of *truth values* (1 standing for *true* and 0 for *false*). We assume that $\phi_R = \cup_{S \in \mathcal{S}} \phi_{RS}$, so that, for every $\alpha \in \phi_R$, at least one state S exists such that $\alpha \in \phi_{RS}$. Then, for every $S \in \mathcal{S}$, let us extend σ_S to the set $\psi_{RS} \subseteq \psi_R$ of all rfs which contain only atomic formulas that belong to ϕ_{RS} , following the standard truth rules of classical propositional logic. We call *assignment function* this extension, denote it by σ_S^e , and call *semantic interpretation* of \mathcal{L}^P the family $\{\sigma_S^e\}_{S \in \mathcal{S}}$. We stress that we do not assume that ϕ_{RS} is a proper subset of ϕ_R : if $\phi_{RS} = \phi_R$, then $\psi_{RS} = \psi_R$ and σ_S^e reduces to a classical assignment function. In general, however, $\{\sigma_S^e\}_{S \in \mathcal{S}}$ can be considered as a *weakened classical semantics* for \mathcal{L}^P .

Pragmatic interpretation. Whenever a semantic interpretation $\{\sigma_S^e\}_{S \in \mathcal{S}}$ is given, a *pragmatic interpretation* of \mathcal{L}^P is defined as a family $\{\pi_S\}_{S \in \mathcal{S}}$, where π_S is a *pragmatic evaluation function* which maps ψ_A onto the set $\{J, U\}$ of *justification values* (J standing for *justified* and U for *unjustified*). We assume that each π_S satisfies the following *justification rules* (where α and δ play the role of metalinguistic variables), which refer to $\{\sigma_S^e\}_{S \in \mathcal{S}}$ and are based on the

informal properties of the metalinguistic concept of proof in natural languages.

JR₁. Let $S \in \mathcal{S}$ and $\alpha \in \psi_R$. Then, $\pi_S(\vdash \alpha) = J$ if $\alpha \in \phi_{RS}$ and a proof exists that $\sigma_S^e(\alpha) = 1$, $\pi_S(\vdash \alpha) = U$ otherwise.

JR₂. Let $S \in \mathcal{S}$ and $\delta \in \psi_A$. Then, $\pi_S(N\delta) = J$ if a proof exists that it is impossible to prove that $\pi_S(\delta) = J$, $\pi_S(N\delta) = U$ otherwise.

JR₃. Let $S \in \mathcal{S}$ and $\delta_1, \delta_2 \in \psi_A$. Then,

- (i) $\pi_S(\delta_1 K \delta_2) = J$ iff $\pi_S(\delta_1) = J$ and $\pi_S(\delta_2) = J$,
- (ii) $\pi_S(\delta_1 A \delta_2) = J$ iff $\pi_S(\delta_1) = J$ or $\pi_S(\delta_2) = J$,
- (iii) $\pi_S(\delta_1 C \delta_2) = J$ iff a proof exists that $\pi_S(\delta_2) = J$ whenever $\pi_S(\delta_1) = J$,
- (iv) $\pi_S(\delta_1 E \delta_2) = J$ iff $\pi_S(\delta_1 C \delta_2) = J$ and $\pi_S(\delta_2 C \delta_1) = J$.

Let us add some terminology and comments on JR₁-JR₃.

First of all, for every $S \in \mathcal{S}$ and $\delta \in \psi_A$ we briefly say in the following that δ is *justified* (*unjustified*) in S whenever $\pi_S(\delta) = J$ (U).

Secondly, let us recall that rules JR₂, JR₃ (iii) and JR₃ (iv) make reference to a notion of proof that belongs to a higher logical level with respect to the notion of proof involved in rules JR₁, JR₃ (i) and JR₃ (ii) [Dalla Pozza and Garola, 1995]. To make this point clear, let us concentrate on JR₂ (JR₃ (iii) and JR₃ (iv) will not be needed in the following) and let us consider an example. Let $S \in \mathcal{S}$ and $\alpha \in \phi_{RS}$. Then, stating that $\vdash \alpha$ is unjustified means that we do not possess any proof of α , but does not prohibit that a proof of α can be produced: hence it does not imply that $N \vdash \alpha$ is justified. The $\text{af } N \vdash \alpha$ is instead justified iff a proof exists that a proof of α cannot be produced: hence, in particular, if a proof exists that α is false. Thus, $\pi_S(N \vdash \alpha) = J$ implies $\pi_S(\vdash \alpha) = U$, but the converse implication does not hold.

Thirdly, let us note that the following *correctness criterion* holds in \mathcal{L}^P because of JR₁.

CC. Let $S \in \mathcal{S}$ and $\alpha \in \psi_{RS}$. Then, $\pi_S(\vdash \alpha) = J$ implies $\sigma_S^e(\alpha) = 1$.

4 The quantum pragmatic language \mathcal{L}_Q^P

As we have anticipated in Sect. 1, we aim to pick out in this section a sublanguage of \mathcal{L}^P and specify the notion of proof as empirical proof in quantum mechanics. For the sake of clearness we will proceed by steps.

4.1 Alphabet and formation rules

The *quantum pragmatic language* \mathcal{L}_Q^P is the sublanguage of \mathcal{L}^P defined by the following syntactic restrictions.

R₁. The set ψ_R^Q of all rfs of \mathcal{L}_Q^P is the subset $\phi_R = \cup_{S \in \mathcal{S}} \phi_{RS}$ of atomic rfs of \mathcal{L}^P .

R₂. The set ψ_A^Q of all afs of \mathcal{L}_Q^P is the set of all afs of \mathcal{L}^P in which only rfs in ψ_R^Q and the logical-pragmatic signs \vdash , N and K occur.

Because of R_1 and R_2 , the set ψ_A^Q of afs of \mathcal{L}_Q^P is made up by all formulas constructed by means of the following recursive rules.

- (i) Let $\alpha \in \psi_R^Q$. Then $\vdash \alpha \in \psi_A^Q$.
- (ii) Let $\delta \in \psi_A^Q$. Then $N\delta \in \psi_A^Q$.
- (iii) Let $\delta_1, \delta_2 \in \psi_A^Q$. Then, $\delta_1 K \delta_2 \in \psi_A^Q$.

The restrictions expressed by R_1 and R_2 are obviously introduced to make it possible to contrive an *intended interpretation* of \mathcal{L}_Q^P in terms of quantum physics. In order to justify R_1 , R_2 and our further assumptions on \mathcal{L}_Q^P , let us discuss this interpretation in some details.

4.2 The intended interpretation of \mathcal{L}_Q^P .

Let Ω be a physical system, characterized in quantum mechanics by a set \mathcal{E} of (first order) physical properties, let \mathcal{U} be a set of individual objects and let $\mathcal{P}(\mathcal{U})$ be the power set of \mathcal{U} . For every $E \in \mathcal{E}$ and $a \in \mathcal{U}$, we write $E(a)$ to formalize the informal sentence “the individual object a has the physical property E_α ” (to avoid proliferation of symbols we do not distinguish here between a physical property and the first order predicate expressing it).

The set \mathcal{S} of states of \mathcal{L}_Q^P can be interpreted as the set of all *quantum states* of some standard realistic interpretation of quantum mechanics, or as the set of all *value states* of some modal interpretation, or with the set of all *microscopic states* in the ESR model (Sect. 2). We do not specify one of these interpretations at this stage, for we wish our treatment to be as general as possible.

For every interpretation of \mathcal{S} we introduce a mapping

$$ext : S \in \mathcal{S} \longrightarrow extS \in \mathcal{P}(\mathcal{U})$$

such that $\{extS\}_{S \in \mathcal{S}}$ is a partition of \mathcal{U} , and say that $extS$ is the *extension* of S (of course, $extS$ depends on the interpretation of S that has been chosen). Furthermore, for every $a \in \mathcal{U}$ we say that “ a is in the state S ” whenever $a \in extS$. Then we provide a physical interpretation of the rfs of \mathcal{L}_Q^P by assuming that a bijective mapping exists

$$I : \alpha \in \psi_R^Q \longrightarrow E_\alpha \in \mathcal{E}$$

such that the following semantic condition holds.

SC. Let $\alpha \in \psi_R^Q$, $S \in \mathcal{S}$, and let $a \in extS$. Then, $\alpha \in \psi_{RS}^Q = \phi_{RS}$ and $\sigma_S(\alpha) = 1$ (0) if and only if the sentence $E_\alpha(a)$ is true (false) according to the interpretation of S that is adopted.

The semantic condition SC introduces different semantic interpretations of \mathcal{L}_Q^P , depending on the interpretation of S that has been chosen. In any case SC implies that $E_\alpha(a)$ takes the same truth value for every individual object in the state S .

The intended interpretation of the afs of \mathcal{L}_Q^P is now immediate if the term *proof* in JR_1 - JR_3 is meant as *empirical proof*, that is, a proof following from quantum mechanics, which is the physical theory that is considered in this paper.

Bearing in mind the intended interpretation above and footnote 2, it is apparent that R_1 in Sect. 4.1 is introduced to select only rfs that have a truth value for some state S and can be interpreted as *testable*, or *verifiable*, sentences, i.e., sentences such that physical procedures exist which test their truth value (which does not always occur, because of incompatibility of properties, in the case of nonatomic, or *molecular*, rfs; note that a similar restriction has been introduced by Dalla Pozza and Garola [1995] when recovering intuitionistic propositional logic within \mathcal{L}^P). R_2 is introduced instead for the sake of simplicity, since only the pragmatic connectives N and K are relevant to our goals in this paper.

4.3 The semantics of \mathcal{L}_Q^P

The semantics of \mathcal{L}_Q^P is obtained by restricting the assignment functions defined on ψ_R to $\psi_R^Q = \phi_R$. Hence, for every $S \in \mathcal{S}$, the assignment function σ_S^e reduces to σ_S and its domain $\psi_{RS}^Q \subseteq \psi_R^Q$ coincides with ϕ_{RS} . Moreover, the semantic condition SC requires us to add the following semantic principle.

SP. *Every assignment function σ_S defined on ψ_R^Q must preserve the truth values and the relations among truth values of rfs of \mathcal{L}_Q^P established by the laws of quantum mechanics via the intended interpretation of \mathcal{L}_Q^P .*

To illustrate SP let us supply an example.

Let $\alpha_1, \alpha_2 \in \psi_R^Q$, $S \in \mathcal{S}$, and for every $a \in \text{ext}S$ let the laws of quantum mechanics imply that $E_{\alpha_2}(a)$ is true whenever $E_{\alpha_1}(a)$ is true. Then, σ_S must be such that, if $\alpha_1 \in \psi_{RS}^Q$ and $\sigma_S(\alpha_1) = 1$, then $\alpha_2 \in \psi_{RS}^Q$ and $\sigma_S(\alpha_2) = 1$.

The semantic principle SP does not provide, however, any explicit rule for establishing whether a rf $\alpha \in \psi_R^Q$ has a truth value in a given state S , and whether this value is *true* or *false*. To make SP more explicit, let us take into account the possible interpretations of the set S mentioned in Sect. 4.2.

Let us firstly interpret S on the set of all *pure* quantum states³ and let us discuss in some details the quantum partial truth assignment introduced in Sect. 2. Let us consider a state $S \in \mathcal{S}$ and a sentence of the form $E(a)$, with $E \in \mathcal{E}$ and $a \in \text{ext}S$. As we have seen in Sect. 2, one assigns a truth value to $E(a)$ which is independent of the measurement context if E is objective in the quantum state S , that is, if the probability $p(E, S)$ associated by quantum mechanics to the pair (E, S) is 1 (truth value *true*) or 0 (truth value *false*). Therefore, let us consider the set $\mathcal{E}_S = \mathcal{E}_S^T \cup \mathcal{E}_S^F$, with $\mathcal{E}_S^T = \{E \mid E \in \mathcal{E}, p(E, S) = 1\}$ and $\mathcal{E}_S^F = \{E \mid E \in \mathcal{E}, p(E, S) = 0\}$. Bearing in mind the mapping I introduced in Sect. 4.2 and condition SC, we obtain that, for every $\alpha \in \psi_R^Q$, $\alpha \in \psi_{RS}^Q$ iff $I(\alpha) \in \mathcal{E}_S$, and $\sigma_S(\alpha) = 1$ ($\sigma_S(\alpha) = 0$) iff $I(\alpha) \in \mathcal{E}_S^T$ ($I(\alpha) \in \mathcal{E}_S^F$). Otherwise, $\sigma_S(\alpha)$ is not defined, that is, α does not belong to ψ_{RS}^Q . This specification of the way in which truth values are assigned can be restated in terms of set-theoretical conditions on the assignment functions of the family $\{\sigma_S\}_{S \in \mathcal{S}}$. To this end, let us consider the set \mathcal{E} of all physical properties of a physical system Ω . It is well

³See footnote 2. We consider here and in the following only pure quantum states to avoid the more complicated formalism required to deal with mixtures.

known that in quantum mechanics \mathcal{E} is the support of a lattice structure $\mathcal{L}(\mathcal{E}) = (\mathcal{E}, \perp, \mathbb{M}, \mathbb{U})$, usually called *standard (sharp) quantum logic*. In this logic \perp , \mathbb{M} and \mathbb{U} are considered as quantum logical connectives. The symbol \perp denotes an involutory unary operation on \mathcal{E} called *orthocomplementation*. The symbols \mathbb{M} and \mathbb{U} denote join and meet, respectively, in $\mathcal{L}(\mathcal{E})$ ⁴. This lattice is orthomodular but not distributive (it also has some further mathematical properties that do not interest us here) [Beltrametti and Cassinelli, 1981]. Moreover, quantum mechanics associates a subset $\mathcal{S}_E \subseteq \mathcal{S}$ of states with every $E \in \mathcal{E}$ in such a way that the set $\mathcal{S}_{\mathcal{E}} = \{\mathcal{S}_E \mid E \in \mathcal{E}\}$, partially ordered by the set inclusion \subseteq , is a lattice $\mathcal{L}(\mathcal{S}_{\mathcal{E}}) = (\mathcal{S}_{\mathcal{E}}, \subseteq) = (\mathcal{S}_{\mathcal{E}}, \perp, \mathbb{M}, \mathbb{U})$ isomorphic to $\mathcal{L}(\mathcal{E})$, and the following properties hold.

- (i) For every $E \in \mathcal{E}$,
 $\mathcal{S}_{E^\perp} = \mathcal{S}_E^\perp$.
- (ii) Let \cap and \cup denote set theoretical intersection and union, respectively. Then for every $E, F \in \mathcal{E}$,
 $\mathcal{S}_{E \mathbb{M} F} = \mathcal{S}_E \mathbb{M} \mathcal{S}_F = \mathcal{S}_E \cap \mathcal{S}_F$,
 $\mathcal{S}_{E \mathbb{U} F} = \mathcal{S}_E \mathbb{U} \mathcal{S}_F \supseteq \mathcal{S}_E \cup \mathcal{S}_F$.

The truth assignments introduced above can now be restated in set-theoretical terms by referring to the lattice $\mathcal{L}(\mathcal{S}_{\mathcal{E}})$. Indeed it can be shown in quantum mechanics that, for every $E \in \mathcal{E}$ and $S \in \mathcal{S}$, $E \in \mathcal{E}_S^T$ (\mathcal{E}_S^F) if and only if $S \in \mathcal{S}_E$ (\mathcal{S}_E^\perp) [Garola and Sozzo, 2013]. Hence, for every $a \in \text{ext}\mathcal{S}$, a truth value of $E(a)$ is defined if and only if $S \in \mathcal{S}_E \cup \mathcal{S}_E^\perp$, which is *true* if and only if $S \in \mathcal{S}_E$, *false* if and only if $S \in \mathcal{S}_E^\perp$. This result can be transformed into an explicit rule for any $\sigma_S \in \{\sigma_S\}_{S \in \mathcal{S}}$, as follows.

TR. Let $\alpha \in \psi_R^Q$ and $S \in \mathcal{S}$. Then, $\alpha \in \psi_{RS}^Q$ if and only if $S \in \mathcal{S}_{E_\alpha} \cup \mathcal{S}_{E_\alpha}^\perp$, and $\sigma_S(\alpha) = 1$ (0) if and only if $S \in \mathcal{S}_{E_\alpha}$ ($S \in \mathcal{S}_{E_\alpha}^\perp$).

The truth rule TR provides a set-theoretical semantic interpretation $\{\sigma_S\}_{S \in \mathcal{S}}$ of \mathcal{L}_Q^P that follows from the general principle SP whenever \mathcal{S} is interpreted as the set of all quantum states of some standard realistic interpretation of quantum mechanics.

Let us now interpret \mathcal{S} as the set of all value states in a modal interpretation of quantum mechanics (Sect. 2) and let us consider the semantic interpretation $\{\sigma_S\}_{S \in \mathcal{S}}$ in this case. Let $a \in \mathcal{U}$ and let us put $\mathcal{E}(a) = \{E(a) \mid E \in \mathcal{E}\}$. Then, a belongs to the extension of some value state and the subset of all sentences of $\mathcal{E}(a)$ that have a truth value is generally broader than the set of all sentences of $\mathcal{E}(a)$ that have a truth value according to the semantic interpretation discussed above. Moreover, whenever truth values of $E(a) \in \mathcal{E}(a)$ are assigned by both interpretations, they coincide. But it must be stressed that quantum mechanics can predict the truth value of $E(a)$ if and only if $E(a)$ has a truth value according

⁴We introduce here the symbols \mathbb{M} and \mathbb{U} in place of the symbols \wedge and \vee that are usually introduced in quantum logic to denote meet and join, respectively, in the lattice $\mathcal{L}(\mathcal{E})$. We indeed want to avoid the (mis)interpretation of these symbols as classical “and” and “or”, respectively.

to the interpretation of \mathcal{S} as a set of quantum states (which correspond to the *dynamical states* in the modal interpretation of quantum mechanics).

Finally, let us interpret \mathcal{S} as the set of all microscopic states in the ESR model (Sect. 2). In this case the semantic interpretation $\{\sigma_S\}_{S \in \mathcal{S}}$ is such that, for every $S \in \mathcal{S}$, $\psi_{RS} = \psi_R$: hence it is a classical semantics for \mathcal{L}_Q^P . If one considers an individual object $a \in \mathcal{U}$, then a belongs to the extension of some microscopic state and all sentences of the set $\mathcal{E}(a)$ defined above have a truth value. Moreover, the truth value of a sentence $E(a) \in \mathcal{E}(a)$ coincides with the value of $E(a)$ according to the interpretation of \mathcal{S} as a set of quantum states (which correspond to the *macroscopic states* in the ESR model) whenever the latter value is assigned. But it must be stressed that also in this case quantum mechanics predicts the truth value of $E(a)$ if and only if $E(a)$ has a truth value according to the latter interpretation of \mathcal{S} .

4.4 The pragmatics of \mathcal{L}_Q^P

Proceeding as in Sect. 4.3, the pragmatics of \mathcal{L}^P is obtained by restricting each pragmatic evaluation functions π_S defined on ψ_A to ψ_A^Q (this restriction will still be denoted by π_S to avoid proliferation of symbols). Moreover, in the case of elementary afs of \mathcal{L}_Q^P we specify the notion of justification as empirical proof by introducing the following pragmatic principle.

PP. *Let $S \in \mathcal{S}$ and $\alpha \in \psi_R^Q$. Then $\pi_S(\vdash \alpha) = J$ if $\alpha \in \psi_{RS}^Q$ and the laws of quantum mechanics allow to prove, via intended interpretation, that $\sigma_S(\alpha) = 1$, $\pi_S(\vdash \alpha) = U$ otherwise.*

The pragmatic principle PP entails that only the interpretation of \mathcal{S} as a set of quantum states is relevant at a pragmatic level. Indeed, it asserts that an elementary af $\vdash \alpha$ is justified if and only if the sentence $E_\alpha(a)$ (with a an individual object in the state S), which corresponds to the atomic rf α via the mapping I (Sect. 4.2), can be proven to have truth value *true*. But such a proof is possible only if the truth value of $E_\alpha(a)$ is assigned by a quantum partial truth assignment, that is, by the truth assignment associated with a quantum state (Sect. 2). Quantum mechanics indeed can predict only these truth values (Sect. 4.3).

Bearing in mind the above remark, we restrict to the interpretation of \mathcal{S} as a set of quantum states in the following, and no further mention of value states or microscopic states will be done.⁵

The notion of justification as empirical proof then extends to afs of ψ_A^Q that are not elementary via rules JR₂ and JR₃.

⁵The notion of proof specified by PP is empirical in the sense that a proof requires the use of physical laws. However, it can be considered empirical also in a different sense, because the same proof can be obtained by means of measurements. It can be shown in fact that a quantum partial truth assignment assigns a value *true* (*false*) to a sentence $E(a)$, with a in the state S , if and only if one can perform a measurement of E on a without modifying S [Garola and Sozzo, 2004].

The pragmatic principle PP does not provide, however, any explicit rule for establishing whether the elementary $\text{af} \vdash \alpha$ is justified or unjustified in a given state S . To make PP more explicit, let us show that the intended interpretation in Sect. 4.2 and PP imply that the pragmatics of \mathcal{L}_Q^P can be expressed in set-theoretical terms. To this end, and bearing in mind the symbols introduced in Sects. 4.2 and 4.3, we restate PP as follows.

PP'. Let $S \in \mathcal{S}$ and let $\alpha \in \psi_R^Q$. Then, $\pi_S(\vdash \alpha) = J$ if $S \in \mathcal{S}_{E_\alpha}$, $\pi_S(\vdash \alpha) = U$ if $S \notin \mathcal{S}_{E_\alpha}$.

The justification rule PP' specifies π_S on the set of all elementary afs of \mathcal{L}_Q^P and constitutes the starting point for our task. In fact, we can now introduce a mapping

$$f : \delta \in \psi_A^Q \longrightarrow \mathcal{S}_\delta \in \mathcal{L}(\mathcal{S}_\mathcal{E}),$$

which associates a *pragmatic extension* \mathcal{S}_δ with every assertive formula $\delta \in \psi_A^Q$, defining f by means of the following recursive rules.

- (i) Let $\alpha \in \psi_R^Q$. Then, $f(\vdash \alpha) = \mathcal{S}_{\vdash \alpha} = \mathcal{S}_{E_\alpha}$.
- (ii) For every $\delta \in \psi_A^Q$, $f(N\delta) = \mathcal{S}_{N\delta} = \mathcal{S}_\delta^\perp$.
- (iii) For every $\delta_1, \delta_2 \in \psi_A^Q$, $f(\delta_1 K \delta_2) = \mathcal{S}_{\delta_1 K \delta_2} = \mathcal{S}_{\delta_1} \mathbin{\mathbb{M}} \mathcal{S}_{\delta_2}$.

The pragmatic evaluation function π_S can then be calculated by assuming the following recursive justification rules.

JR₁^Q. Let $S \in \mathcal{S}$ and $\alpha \in \psi_R^Q$. Then, $\pi_S(\vdash \alpha) = J$ if $S \in \mathcal{S}_{\vdash \alpha}$, $\pi_S(\vdash \alpha) = U$ if $S \notin \mathcal{S}_{\vdash \alpha}$.

JR₂^Q. Let $S \in \mathcal{S}$ and $\delta \in \psi_A^Q$. Then, $\pi_S(N\delta) = J$ if $S \in \mathcal{S}_{N\delta}$, $\pi_S(N\delta) = U$ if $S \notin \mathcal{S}_{N\delta}$.

JR₃^Q. Let $S \in \mathcal{S}$ and $\delta_1, \delta_2 \in \psi_A^Q$. Then, $\pi_S(\delta_1 K \delta_2) = J$ if $S \in \mathcal{S}_{\delta_1 K \delta_2}$, $\pi_S(\delta_1 K \delta_2) = U$ if $S \notin \mathcal{S}_{\delta_1 K \delta_2}$.

Rules JR₁^Q-JR₃^Q specialize JR₁-JR₃, respectively, in our present context. They are suggested by the following arguments.

Rule JR₁^Q. From PP' and (i).

Rule JR₂^Q. Let $\alpha \in \psi_{RS}^Q$. Then we must consider three alternatives, that is, $S \in \mathcal{S}_{\vdash \alpha}$, $S \in \mathcal{S}_{\vdash \alpha}^\perp$ and $S \notin \mathcal{S}_{\vdash \alpha} \cup \mathcal{S}_{\vdash \alpha}^\perp$. If $S \in \mathcal{S}_{\vdash \alpha}$, hence $S \notin \mathcal{S}_{\vdash \alpha}^\perp = \mathcal{S}_{N(\vdash \alpha)}$, then $\vdash \alpha$ is justified in S because of JR₁^Q: therefore no proof exists that $\vdash \alpha$ cannot be justified, which implies that $N(\vdash \alpha)$ is unjustified. If $S \in \mathcal{S}_{\vdash \alpha}^\perp = \mathcal{S}_{N(\vdash \alpha)}$, then $S \in \mathcal{S}_{E_\alpha}^\perp$ because of (i), hence α is false ($\sigma_S(\alpha) = 0$) because of the TR rule (Sect. 4.3): it follows that a proof exists that $\vdash \alpha$ cannot be justified in S , which implies that $N(\vdash \alpha)$ is justified in S . If $S \notin \mathcal{S}_{\vdash \alpha} \cup \mathcal{S}_{\vdash \alpha}^\perp$, then α has no truth value according to the TR rule, that is, according to the quantum truth assignment. Nevertheless a value can be *actualized* by suitable measurements, and it can be *true*, or α can have a truth value according to the modal interpretation of quantum mechanics and it has a truth value in the ESR model. Hence no proof of α is supplied by quantum mechanics, but no proof exists that $\vdash \alpha$ cannot be justified in S , which implies that $N(\vdash \alpha)$ is unjustified in S .

Rule JR_3^Q . Let $\alpha_1, \alpha_2 \in \psi_{RS}^Q$. Then the af $(\vdash \alpha_1)K(\vdash \alpha_2)$ is justified if and only if $\vdash \alpha_1$ and $\vdash \alpha_2$ are justified, that is, if and only if $S \in \mathcal{S}_{\vdash \alpha_1}$ and $S \in \mathcal{S}_{\vdash \alpha_2}$: hence, if and only if $S \in \mathcal{S}_{\vdash \alpha_1} \cap \mathcal{S}_{\vdash \alpha_2} = \mathcal{S}_{\vdash \alpha_1} \mathbin{\mathbb{M}} \mathcal{S}_{\vdash \alpha_2} = \mathcal{S}_{(\vdash \alpha_1)K(\vdash \alpha_2)}$.

5 The pragmatic interpretation of quantum logic

Bearing in mind the definition of the mapping f in Sect. 4.4, the justification rules JR_1^Q - JR_3^Q can be unified by the following rule.

JR^Q . Let $S \in \mathcal{S}$ and $\delta \in \psi_A^Q$. Then, $\pi_S(\delta) = J$ if $S \in \mathcal{S}_\delta$, $\pi_S(\delta) = U$ if $S \notin \mathcal{S}_\delta$.

Based on JR^Q , one can introduce a preorder (binary, transitive) relation \prec on ψ_A^Q .

OR. Let $\delta_1, \delta_2 \in \psi_A^Q$. Then, $\delta_1 \prec \delta_2$ if and only if, for every $S \in \mathcal{S}$, $\pi_S(\delta_1) = J$ implies $\pi_S(\delta_2) = J$.

It follows from JR^Q and OR that $\delta_1 \prec \delta_2$ if and only if $\mathcal{S}_{\delta_1} \subseteq \mathcal{S}_{\delta_2}$. It is then apparent that the mapping f is an order homomorphism of (ψ_A^Q, \prec) onto $(\mathcal{S}_\mathcal{E}, \subseteq)$. Since $f(N\delta) = \mathcal{S}_\delta^\perp$ and $f(\delta_1 K \delta_2) = \mathcal{S}_{\delta_1} \mathbin{\mathbb{M}} \mathcal{S}_{\delta_2}$, we briefly say that f makes the connectives N and K correspond to the lattice operations $^\perp$ and $\mathbin{\mathbb{M}}$, respectively. Furthermore, let us introduce a derived connective A^Q in \mathcal{L}_Q^P , defined by the equation

$$\delta_1 A^Q \delta_2 = N((N\delta_1)K(N\delta_2)).$$

Then we get

$$f(\delta_1 A^Q \delta_2) = (\mathcal{S}_{(N\delta_1)K(N\delta_2)})^\perp = (\mathcal{S}_{\delta_1}^\perp \mathbin{\mathbb{M}} \mathcal{S}_{\delta_2}^\perp)^\perp = \mathcal{S}_{\delta_1} \mathbin{\mathbb{U}} \mathcal{S}_{\delta_2}.$$

Hence, f makes the connective A^Q correspond to the lattice operation \mathbb{U} . Thus, our homomorphism shows that the quantum logical connectives $^\perp$, $\mathbin{\mathbb{M}}$ and \mathbb{U} can bear an interpretation as logical-pragmatic signs rather than logical-semantic signs.

The above interpretation can be made more cogent by introducing an equivalence relation \approx on ψ_A^Q , defined as follows.

ER. Let $\delta_1, \delta_2 \in \psi_A^Q$. Then, $\delta_1 \approx \delta_2$ if and only if $\delta_1 \prec \delta_2$ and $\delta_2 \prec \delta_1$ (equivalently, for every $S \in \mathcal{S}$, $\pi_S(\delta_1) = J$ if and only if $\pi_S(\delta_2) = J$).

JR^Q and ER imply indeed that $\delta_1 \approx \delta_2$ if and only if $\mathcal{S}_{\delta_1} = \mathcal{S}_{\delta_2}$. Let us consider the quotient set $\psi_A^{Q'} = \psi_A^Q / \approx$ and the relation \prec' canonically induced on $\psi_A^{Q'}$ by the relation \prec defined on ψ_A^Q . Then, \prec' is a partial order (binary, transitive, antisymmetric and reflexive) on $\psi_A^{Q'}$. Moreover, the mapping f induces an order isomorphism of $(\psi_A^{Q'}, \prec')$ onto $(\mathcal{S}_\mathcal{E}, \subseteq)$. Hence $(\psi_A^{Q'}, \prec')$ is an (orthomodular) lattice, in which lattice operations N' , K' and $A^{Q'}$ are defined which correspond to the operations $^\perp$, $\mathbin{\mathbb{M}}$ and \mathbb{U} defined on $\mathcal{S}_\mathcal{E}$, respectively. These operations are related to the connectives N , K and A^Q by the following equations.

$$N'[\delta]_\approx = [N\delta]_\approx,$$

$$\begin{aligned} [\delta_1]_{\approx} K' [\delta_1]_{\approx} &= [\delta_1 K \delta_1]_{\approx}, \\ [\delta_1]_{\approx} A^{Q'} [\delta_1]_{\approx} &= [\delta_1 A^Q \delta_1]_{\approx}. \end{aligned}$$

We have thus obtained a pragmatic structure $(\psi_A^{Q'}, \prec') = (\psi_A^{Q'}, N', K', A^{Q'})$ which is isomorphic to the quantum logic $(\mathcal{S}_{\mathcal{E}}, \subseteq) = (\mathcal{S}_{\mathcal{E}}, \perp, \sqcap, \sqcup)$ introduced in Sect. 4.3. These two structures can then be identified.

6 Conclusions

The results stated at the end of the preceding section are philosophically important. They imply that QL can be considered as a pragmatic structure, formalizing the properties of empirical justification in quantum mechanics rather than a notion of quantum truth specific of this theory. This conclusion agrees with the general integrationist perspective (global pluralism) mentioned in Sect. 1, according to which nonstandard logics are interpreted as theories of metalinguistic notions different from truth, thus avoiding incompatibility with classical notions and preserving the globality of logic. Our aims in Sect. 1 are thus reached. Of course this achievement has a price. Indeed, if one adopts a standard realistic or a modal interpretation of quantum mechanics, one can reconcile QL with a classical notion of truth at the expense of weakening this notion by introducing partial truth assignments. A complete reconciliation of QL with classical logic is possible only by accepting the reinterpretation of quantum probabilities introduced by the ESR model. In any case, our results provide a further example of the explanatory power of the (generalized) pragmatic extension \mathcal{L}^P of classical logic, in which different logical systems may coexist without conflicting because they are interpreted as formalizing different metalinguistic concepts.

It remains to observe that conclusions similar to ours have been drawn in [Garola, 1992, 2008] and [Garola and Sozzo, 2013]. In particular, in the last of these papers a classical predicate calculus $L(x)$ is constructed and enriched by introducing a physical preorder (which is implied by the logical order but generally does not coincide with it) induced by the theory-dependent notion of “certainly true in a state S ”. A structure isomorphic to QL is then recovered by selecting a subset of sentences of $L(x)$ that are verifiable according to quantum mechanics and adding a physically justified orthocomplementation.⁶ Our present approach, however, is more general in several senses. Firstly, it is constructed in such a way to allow an orthodox physical interpretation of the truth values that are assigned to the radical formulas of \mathcal{L}^P . On the contrary, the approach in [Garola and Sozzo, 2013] introduces a classical semantics on

⁶It is noteworthy that similar procedures also lead to recover classical Boolean structures whenever the language of classical mechanics is considered, which implies that the logical and the physical order coincide and all sentences are assumed to be verifiable. Moreover structures isomorphic to QL can be obtained in this classical case if the notion of verification is suitably restricted, so that only some sentences turn out to be verifiable. This result proves that QL occurs because of the notion of verification that is adopted and does not characterize quantum mechanics, consistently with a known position of some scholars concerned with the foundations of quantum mechanics [Aerts, 1988, 1991, 1995, 1998, 1999].

$L(x)$ which has a physical meaning only if one adopts the reinterpretation and generalization of quantum mechanics propounded by the ESR model. Secondly, the procedures in [Garola and Sozzo, 2013] are, paraphrasing Salmon’s classification of scientific explanations [Salmon, 1989], “bottom-up”, because they explain QL in terms of basic logical and physical structures. On the contrary, our present procedures are “up-down”. Indeed, they are based on the (generalized) language \mathcal{L}^P , which has a pragmatic interpretation that does not depend on specific physical theories and is suitable for recovering different non-standard logics by specifying different notions of proof. Hence our interpretation of QL as a pragmatic structure constitutes an instantiation of a general method in a special case (empirical quantum proof).

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BIBLIOGRAPHY

- Aerts, D. (1988). The physical origin of the EPR paradox and how to violate-Bell inequalities by macroscopic systems. In Lahti, P., et al. (Eds.), *Symposium on the foundations of modern physics* (pp. 305-320). Singapore: World Scientific.
- Aerts, D. (1991). A macroscopic classical laboratory situation with only macroscopic classical entities giving rise to a quantum mechanical probability model. In Accardi, L. (Ed.), *Quantum probability and related topics* (pp. 75-85). Singapore: World Scientific.
- Aerts, D. (1995). Quantum structures: An attempt to explain their appearance in nature. *International Journal of Theoretical Physics*, 34, 1165-1186.
- Aerts, D. (1998). The hidden measurement formalism: What can be explained and where quantum paradoxes remain. *International Journal of Theoretical Physics*, 37, 291-304.
- Aerts, D. (1999). Quantum mechanics: Structures, axioms and paradoxes. In Aerts, D. & Pykacz, J. (Eds.), *Quantum physics and the nature of reality* (pp. 141-205). Dordrecht: Kluwer.
- Bell, J. S. (1964). On the Einstein-Podolski-Rosen Paradox. *Physics* 1, 195-200.
- Bell, J. S. (1966). On the Problem of Hidden Variables in Quantum Mechanics. *Review of Modern Physics*, 38, 447-452.
- Bellin, G. (2014). Categorical proof theory of co-intuitionistic linear logic. *Logical Methods in Computer Science*, 10, 1–36. www.lmcs-online.org.
- Bellin, G. (2015). Assertions, hypotheses, conjectures, expectations: rough-sets semantics and proof-theory. In Pereira, L. C., et al. (Eds), *Advances in*

Natural Deduction. A Celebration of Dag Prawitz's Work, Trends in Logic, 39 (pp. 193–241). Springer Science+Business Media.

Bellin, G. and Biasi, C. (2004). Towards a logic for pragmatics. Assertions and conjectures. *Journal of Logic and Computation*, 14, 473–506.

Bellin, G., Carrara, M., Chiffi, D. (2015b) On an intuitionistic logic for pragmatics. *Journal of Logic and Computation*, DOI: 10.1093/logcom/exv036.

Bellin, G., Carrara, M., Chiffi, D. and Menti, A. (2015a). Pragmatic and dialogic interpretations of bi-intuitionism. Part 1. *Logic and Logical Philosophy*, 23, pp. 449–480, 2014. Errata Corrigé in *Logic and Logical Philosophy*, 2015. Online April 18, 2015.

Bellin, G. & Dalla Pozza, C. (2002). A pragmatic interpretation of substructural logics. In Sieg, W., et al. (Eds.), *Reflections on the foundations of mathematics - Essays in honor of Solomon Feferman* (pp.139-163). *Association for Symbolic Logic, Lecture Notes in Logic*, 15.

Bellin, G. & Ranalter, K. (2003). A Kripke-style semantics for the intuitionistic logic of pragmatics ILP. *Journal of Logic and Computation*, 13 (5), 755-775.

Beltrametti, E. G. & Cassinelli, G. (1981). *The logic of quantum mechanics*. Reading, MA: Addison.

Biasi, C. and Aschieri, F. (2008). A term assignment for polarized bi-intuitionistic logic and its strong normalization. *Fundamenta Informaticae, Special issue on Logic for Pragmatics*, 84, 185–205.

Busch, P., Lahti, P. J. & Mittelstaedt, P. (1996). *The quantum theory of measurement*. Berlin: Springer.

Carnap, R. (1932). Überwindung der Metaphysik durch logische Analyse der Sprache. *Erkenntnis II*, 219-241.

Carnap, R. (1949). Truth and confirmation. In Feigl, H. & Sellars, W. (Eds.), *Readings in philosophical analysis* (pp.119-127). New York: Appleton-Century-Crofts, Inc.

Carrara, M. & Chiffi, D. (2013). The knowability paradox in the light of a logic for pragmatics. In Ciuni, R. et al. (Eds.), *Advances in philosophical logic* (pp. 33-48). *Proceedings of Trends in Logic XI, Studia Logica Library*. Berlin: Springer.

Dalla Chiara, M. L., Giuntini, R. & Greechie, R. (2004). *Reasoning in quantum theory*. Dordrecht, Kluwer.

Dalla Pozza, C. & Garola, C. (1995). A pragmatic interpretation of intuitionistic propositional logic. *Erkenntnis*, 43, 81-109.

Dummett, M. (1977). *Elements of intuitionism*. Oxford: Clarendon Press.

Dummett, M. (1978). *Truth and other enigmas*. London: Duckworth.

Dummett, M. (1979). What does the appeal to use do for the theory of meaning? In Margalit A. (Ed.), *Meaning and use* (pp. 123-135). Dordrecht: Reidel.

Dummett, M. (1980). Comments on professor's Prawitz paper. In Von Wright G. H. (Ed.), *Logic and philosophy/logique et philosophie* (pp. 11-18). Dordrecht: Martinus Nijhoff Publishers.

- Frege, G. (1879). *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle: Nebert.
- Frege, G. (1891). Funktion und Begriff. *Vortrag, gehalten in der Sitzung vom 9. Januar 1891 der Jenaischen Gesellschaft für Medizin und Naturwissenschaft*. Jena: Hermann Pohle. (Collected in Angelelli I. (Ed.), *Frege G.: Kleine Schriften* (pp. 124-142). Hildesheim: Olms, 1967).
- Frege, G. (1893.). *Grundgesetze der Arithmetik 1*. Jena: Pohle (Hildesheim: Olms, 1967).
- Frege, G. (1918). Der Gedanke. Eine Logische Untersuchung. In *Beiträge zur Philosophie des deutschen Idealismus I* (1918-19), pp. 58-77. (Collected in Angelelli I. (Ed.), *Frege G.: Kleine Schriften* (pp. 342-362). Hildesheim: Olms, 1967).
- Garola, C. (1992). Truth versus testability in quantum logic. *Erkenntnis*, 37, 197-222.
- Garola, C. (2008). Physical propositions and quantum languages. *International Journal of Theoretical Physics*, 47, 90-103.
- Garola, C. (2015). A survey of the ESR model for an objective reinterpretation of quantum mechanics. *International Journal of Theoretical Physics*, DOI 10.1007/s10773-015-2618-y.
- Garola, C. & Persano, M. (2014). Embedding quantum mechanics into a broader noncontextual theory. *Foundations of Science*, 19, 217-239. DOI 10.1007/s10699-013-9341-z.
- Garola, C., Persano, M., Pykacz, J. & Sozzo, S. (2014). Finite local models for the GHZ experiment. *International Journal of Theoretical Physics*, 53, 622-644. DOI 10.1007/s10773-013-1851-5.
- Garola, C. & Sozzo, S. (2009). The ESR model: A proposal for a non-contextual and local Hilbert space extension of QM. *Europhysics Letters*, 86, 20009.
- Garola, C. & Sozzo, S. (2010). Embedding quantum mechanics into a broader noncontextual theory: A conciliatory result. *International Journal of Theoretical Physics*, 49, 3101-3117.
- Garola, C. & Sozzo, S. (2011a). Generalized observables, Bell's inequalities and mixtures in the ESR model, *Foundations of Physics*, 41, 424-449.
- Garola, C. & Sozzo, S. (2011b). The modified Bell inequality and its physical implication in the ESR model. *International Journal of Theoretical Physics*, 50, 3787-3799.
- Garola, C. & Sozzo, S. (2011c). Representation and interpretation of quantum mixtures in the ESR model. *Theoretical and Mathematical Physics*, 168, 912-923.
- Garola, C. & Sozzo, S. (2012). Extended representation of observables and states for a noncontextual representation of QM. *Journal of Physics A: Mathematical and Theoretical*, 45, 075303 (13pp).
- Garola, C. & Sozzo, S. (2013). Recovering quantum logic within an extended classical framework. *Erkenntnis*, 78, 399-419.
- Garola, C., Sozzo, S. & Wu, J. (2015). Outline of a generalization and a reinterpretation of quantum mechanics recovering objectivity. *ArXiv:1402.4394v3*

[quant-ph].

- Girard, J-Y. (1987). Linear logic. *Theoretical Computer Science*, 50, 1-102.
- Haak, S. (1978). *Philosophy of logic*. Cambridge: Cambridge University Press.
- Jammer, M. (1974). *The Philosophy of Quantum Mechanics*. New York: Wiley.
- Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. London: Addison-Wesley.
- Kochen, S. and Specker, E. P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematical Mechanics*, 17, 59-87.
- Lombardi, O. and Dieks, D. (2014). Modal interpretations of quantum mechanics. *The Stanford Encyclopedia of Philosophy* (spring 2014 edition), Zalta E. N. (Ed.). URL=<<http://plato.stanford.edu/archives/spr2014/entries/qm-modal/>>.
- Ludwig, G. (1983). *Foundations of Quantum Mechanics I*. New York: Springer.
- Mermin, N. D. (1993). Hidden variables and the two theorems of John Bell. *Reviews of Modern Physics*, 65, 803-815.
- Piron, C. (1976). *Foundations of Quantum Physics*. Reading, MA: Benjamin.
- Popper, K. (1969). *Conjectures and refutations*. London: Routledge and Kegan Paul.
- Prawitz, D. (1977). Meaning and proof: on the conflict between classical and intuitionistic logic. *Theoria*, 43, 1-40.
- Prawitz, D. (1980). Intuitionistic logic: a philosophical challenge. In Von Wright G. H. (Ed.), *Logic and philosophy/Logique et philosophie* (pp. 1-10). Dordrecht: Martinus Nijhoff Publishers.
- Prawitz, D. (1987). Dummett on a theory of meaning and its impact on logic. In Taylor B. M. (Ed), *Michael Dummett* (pp. 117-165). Dordrecht: Martinus Nijhoff Publishers.
- Ranalter, K. (2008). A semantic analysis of a logic for pragmatics with assertions, obligations and causal implication. *Fundamenta Informaticae*, 84 (3-4), 443-470
- Rédei, M. (1998). *Quantum Logic in Algebraic Approach*. Dordrecht: Kluwer.
- Reichenbach, H.: (1947). *Elements of symbolic logic*. New York: The Free Press.
- Russell, B. (1940). *An inquiry into meaning and truth*. London: Allen & Unwin.
- Russell, B. (1950). Logical positivism. *Revue Internationale de Philosophie*, 4, 3-19. (Collected in Russell, B. (1956). *Logic and knowledge*. London; Allen & Unwin).
- Salmon, W.C. (1989). Four decades of scientific explanation. In Kitcher, P. & Salmon, W. C. (Eds.), *Scientific explanation. Minnesota studies on the philosophy of science (13)* (pp. 3-219). Minneapolis: University of Minnesota Press.
- Stenius, E. (1969). Mood and language-games. In Davis, J. W. et al.(Eds.), *Philosophical logic* (pp. 251-271). Dordrecht: Reidel.

Tarski, A. (1933). Pojecie prawdy w jezykach nauk dedukcyjnych. *Acta Towarzystwa Naukowego i Literackiego Warszawskiego*, 34, V-16; (1956). The concept of truth in formalized languages. In Woodger J. M. (Ed.), *Logic, semantics, metamathematics* (pp. 152-268). Oxford: Oxford University Press (trans.).

Tarski, A. (1944). The semantic conception of truth and the foundations of semantics. *Philosophy and phenomenological research*, 4, 341-375 (1952. In Linski, L. (Ed.), *Semantics and the philosophy of language* (pp. 13-47). Urbana: University of Illinois Press.

Timpson, C. (2008). Philosophical aspects of quantum information theory. In Rickler, D. (Ed.), *The Ashgate companion to the new philosophy of physics* (pp. 197-261). Aldershot: Ashgate.

Troelstra, A. and Van Dalen, D. (1988). *Constructivism in mathematics*. Amsterdam: North Holland.

Van Dalen, D. (1986). Intuitionistic logic. In Gabbay, D. & Guenther, F. (Eds.), *Handbook of philosophical logic III* (pp. 225-339). Dordrecht: Reidel.

White, G. (2008). Davidson and Reiter on action. *Fundamenta Informaticae*, 84 (2), 259-289.